Preliminary Statistics September 2013

Answers to class exercise

Problem Set 5: Hypothesis Testing

1. (a) You would have to go through the following steps:

   i. You first have to specify your null and alternative hypothesis with:
   $$H_0: \mu = \mu_0$$
   $$H_1: \mu \neq \mu_0$$
   ii. You would then construct a test statistic with
   $$\tau = \frac{\hat{\mu} - \mu_0}{SE(\hat{\mu})}$$
   Where $\hat{\mu}$ is the sample mean and the standard error is defined by $SE(\hat{\mu}) = s/\sqrt{n}$, with s being the sample variance and n being the sample size.
   iii. Find how the test statistic is distributed: For a large sample size (n > 30) it is distributed as a standard normal (as you standardised it already) and for a small sample size and an unknown variance it is distributed as a t-distribution with n-1 degrees of freedom.
   iv. Fix the significance level of the test (conventionally at 5%; but also 1% and 10% is used) and find the critical value for the distribution of the test statistic.
   v. Apply the decision rule: Reject the null hypothesis if the test statistic lies outside the range of the critical value.

   (b) Type I error is rejecting the null hypothesis when it is true i.e. P(rejecting H_0|H_0 is true). Type II error is accepting the null when it is false i.e. P(accepting H_0|H_0 is true). Above we fixed the probability of Type I error at 5%.
   These two errors have an inverse relationship as Type II error increases with Type I error decreasing and vice versa.

   (c) The probability of Y taking on larger values than the specified value Y_0 i.e. $P(Y > Y_0)$ could be estimated by firstly standardising so that:
   $$P\left(\frac{(Y - \hat{\mu})}{s} > \frac{(Y_0 - \hat{\mu})}{s}\right) = P\left(Z > \frac{(Y_0 - \hat{\mu})}{s}\right)$$
   One can then use the table of the normal distribution to obtain the respective probability (this is because you know that $Y \sim IN(\mu, \sigma^2)$ so that the standardisation of Y yields $Z \sim (0,1)$.

   NOTE: We standardising by dividing through the standard deviation i.e. the square root of the sample variance and not the standard error. This is because we want to get the probability of one realisation of our random variable Y taking on a particular value or in this case a range of values. This is different than estimating the probability of an estimator for the sample mean takes on certain values.

   (d) The confidence interval for μ is defined as:
   $$P(\hat{\mu} - SE(\hat{\mu}) * z^* < \mu < \hat{\mu} + SE(\hat{\mu}) * z^*) = 95\%$$

We know that $\hat{\mu}$ follows or normal distribution (as $Y \sim IN(\mu, \sigma^2)$). Hence we can use the critical values for the standard normal distribution (after standardising):

$$P\left(\hat{\mu} - \frac{\sigma}{\sqrt{n}} * 1.96 < \mu < \hat{\mu} + \frac{\sigma}{\sqrt{n}} * 1.96\right) = 95\%$$

2. (a) We can calculated the respective probabilities with using the information provided:

$$W \sim N(1005,4)$$
$$s = 2$$

Where W – the random variable – is the weight of a bag sugar.
Hence we are looking for:

(i) $P(W < 1000)$
(ii) $P(1004 < W < 1000)$

We can standardise in order to get the critical values for the z-statistic, exploit the symmetry of the statistic and look up the respective probability values in the table:

(i) $P(W < 1000) = P\left(\frac{W-1005}{2} < \frac{1000-1005}{2}\right) = P(Z < -2.5) = P(Z > 2..5) = 0.0062$

(ii) $P(1004 < W < 1006) = P\left(\frac{1004-1005}{2} < \frac{W-1005}{2} < \frac{1006-1005}{2}\right) = P(-0.5 < Z < 0.5) = 1 - 2 * P(Z > 0.5) = 0.383$

(b) We can calculate the standard error of the estimator for the mean $\hat{\mu}_W = 1004$ for $n = 16$ and the probability for getting another sample mean with value below $\hat{\mu}_W$, based on:

(i) Our knowledge about the variance of the estimator for the mean which is given by:

$$Var(\hat{\mu}_W) = \frac{\sigma^2}{n}$$

By simply taking the square root we get the standard error of the mean:

$$SE(\hat{\mu}_W) = \frac{\sigma}{\sqrt{n}} = \frac{2}{4} = 0.5$$

Where σ^2 is the population variance and σ the population standard deviation.

(i) In order to calculate to probability of $P(\hat{\mu}_W \leq 1004)$ one has to standardise the critical values. As W is normally distributed $\hat{\mu}_W$ is also normally distributed (as it is a linear transformation of W). Hence, after standardisation we can use the statistical table for the standard normal distribution to find the respective probability:

$$P\left(\frac{\hat{\mu}_W - 1005}{0.5} \leq \frac{1004 - 1005}{0.5}\right) = P(Z < -2) = P(Z > 2) = 0.0228.$$

NOTE: in (b) we are estimating the probability of the estimator to take on a specific value, while in (a) we were estimating the probability of W to take on a certain value. For the latter case the values of the random variable W have to be standardised by dividing by the population standard error: $s = 2$. For the former case one has to divide the values of the estimator (also a random variable) through the standard error: $SE(\hat{\mu}_W) = 0.5$. These are conceptually very different!

(c) This is actually a hypothesis test, where:

$$\mathrm{H}_0: \mu = 1005$$
$$\mathrm{H}_1: \mu \neq 1005$$

With the test statistic:

$$\tau = \frac{\hat{\mu}_W - \mu_W}{SE(\hat{\mu}_W)} = \frac{1004 - 1005}{0.2} = -2$$

At the 5% significance level the critical value is ± 1.96 (we assume the test statistic is standard normally distributed, since the variance is considered as known). Hence, we reject the null hypothesis and conclude: At the 5% significance level the true population mean is statistically significantly different from 1005.

NOTE: In the previous sub-question we have calculated the probably of the test score, which is multiplied by 2 (2-sided test) which would give the p-value 0.0456 < 0.05. Hence we would reject the null hypothesis.

3. We have the following information provided by the question:
   i. $n = 100$
   ii. $\hat{\mu} = 57$
   iii. $s = 2$
   iv. For testing hypothesis regarding out estimator we would also need to know the standard error: $SE(\hat{\mu}) = \frac{s}{\sqrt{n}} = \frac{2}{10} = 0.2$.

(a) Test for the one-sided hypothesis:

$$\mathrm{H}_0: \mu = 60$$
$$\mathrm{H}_1: \mu > 60$$

The test statistic is given by:

$$\tau = \frac{\hat{\mu} - \mu}{SE(\hat{\mu})} = \frac{57 - 60}{0.2} = -15$$

At the 5% level the critical value is ± 1.645 (you find this by looking at the standard normal statistical table and find the critical value so that the probability of Z being on the left hand side of this critical value is 5%). At the 1% significance level the respective critical value would be ± 2.326.

Hence,

$$P(Z < -1.645) = 0.05 \; or \; P(|Z| > 1.645) = 0.05$$

$$P(Z < -2.326) = 0.05 \; or \; P(|Z| > 2.326) = 0.01$$

At both the 1% and the 5% significance level you would reject the null hypothesis as the absolute value of the test statistic is larger than the right hand side critical values: $|\tau| > 1.645 \; and \; |\tau| > 2.326$.

(b) Test for the two-sided hypothesis:

$$\mathrm{H}_0: \mu = 60$$
$$\mathrm{H}_1: \mu \neq 60$$

The test statistic is given by:

$$\tau = \frac{\hat{\mu} - \mu}{SE(\hat{\mu})} = \frac{57 - 60}{0.2} = -15$$

The test statistic remains the same. However, the critical values are changing. At the 5% level the critical value is ± 1.96 (you find this by looking at the standard normal statistical table and find the critical value so that the probability of Z being on the left hand side or on the right hand side of this critical value and of the negative of the critical value is 2.5%). At the 1% significance level the respective critical value would be ± 2.57.

Hence,

$$P(-1.96 < Z < 1.96) = 1 - 0.05 = 0.95 \; or \; P(|Z| > 1.96) = 0.025 \; or \; P(Z < -1.96 \; and \; Z > 1.96) = 0.05$$

$$P(|Z| > 2.57) = 0.005 \; or \; P(Z < -2.57 \; and \; Z > 2.57) = 0.01$$

At both the 1% and the 5% significance level you would reject the null hypothesis as the absolute value of the test statistic is larger than the right hand side critical values: $|\tau| > 1.96 \; and \; |\tau| > 2.57$.