Preliminary Statistics September 2013

Answers to class exercise

Problem Set 2: Probability Theory

1. (a) If A and B are mutually exclusive there is not intersection of the two events and hence their joint probability is zero: P(A ∩ B) = 0.

   (b) If A and B are independent, the A occurring does not impact the probability of B occurring and vice versa. Hence their joint probability is the simple product of their marginal probabilities: P(A ∩ B) = P(A) * P(B).

   (c) P (A U B) is the probability of either A or B or both occurring. Hence it is the sum of their marginal probabilities. If there is an intersection i.e. the two events are not independent i.e. the joint probability is not zero, on has to subtract the intersection. So that: P (A U B) = P(A) + P(B) if the two are independent and P (A U B) = P(A) + P(B) – P(A ∩ B) if the events are not independent.

   (d) This can be shown by the Bayes Theorem/Rule: P(A|B) = P(A ∩ B)/P(B) and P(B|A) = P(A ∩ B)/P(A). Both dies of the equations can be multiplied by P(B) and P(A) respectively, so that: P(A|B)*P(B) = P(A ∩ B) = P(B|A)*P(A). Hence: P(A|B)*P(B) = P(B|A)*P(A). Dividing both sides by P(B) gives: P(A|B) = [P(B|A)*P(A)]/P(B).

2. This is the St Petersburg paradox. The expected value of getting a head [H] or a tail [T] is 0.5 each. Each time the coin is flipped this is a new game; the outcome of the new game is independent from the outcome of the former game. Hence the joint probability of two successive games resulting in tail is the product of the probabilities of the marginal probability of getting a tail each game i.e. 0.5. The expected payoff is then the sum of all possible payoffs weighted by their respective probability i.e. the payoff from H in the first round weighted by its probability (2£*P(H)) plus the payoff from H in the second round (4£*P(T)P(H)) plus the payoff from H in the third round (5£*P(T)P(T)P(H)), and so forth. Hence:

$$E(game) = \sum_{i=1}^{\infty} payoff_i * f(payoff_i) = 2 * \frac{1}{2} + 2^2 * \frac{1}{2^2} + \cdots + \infty$$

   With i being the number of times the coin is flipped. The expected value is infinite, but few would pay an infinite amount of money to play it. The usual explanation is in terms of the diminishing marginal utility of money, which makes the expected utility of the game less than infinity.

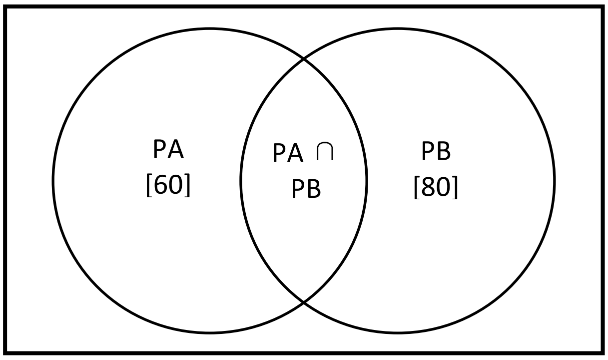
3. Given that P(F) is the marginal probability of a customer being female. P(M) being the marginal probability of a customer being male with P(M) = 1 – P(F) = P(not F). Similarly the marginal probability of a customer purchasing food is given by P(FF) and the respective probability of a customer not purchasing food is given by P(NF) = 1 – P(F) = P(not F).

   Information at hand can be summarised as following:

   (i) P(M) = 0.6 and P(F) = 0.4
   (ii) P(PF) = 0.5 and P(NF) = 0.5
   (iii) P(M ∩ PF) = 0.15

   (a) It is asked for the joint probability of a customer being female and purchasing food, i.e. P(F ∩ PF) ?
   You can solve this using the equation: P(PF) = P(F ∩ PF) + P(M ∩ PF), hence: P(F ∩ PF) = P(PF) – P(M ∩ PF) = 0.5 – 0.15 = 0.35.
   (b) It is asked for the conditional probability of a customer purchasing food given that she is female, i.e. P(PF|F) ?
   You can solve this using the relationship: P(PF|F) = P(PF ∩ F)/P(F) = 0.35/0.4 = 0.875.
   (c) It is asked for the conditional probability of a customer being female given that she purchases food, i.e. P(F|PF) ?
   You can solve this using Bayes Theorem: P(F|PF) = [P(PF|F)*P(F)]/P(PF) = [0.875*0.4]/0.5 = 0.7.

4. (a) Diagrammatical presentation of the information given:



   How to fill in the table:

   - P(FA) = 1 – P(PA) = 1 – 0.6 = 0.4 -> 40 students
   - P(FB) = 1 – P(PB) = 1 – 0.8 = 0.2 -> 20 students
   - P(FA ∩ FB) = 1 – [P(PA) + P(PB) – P(PA ∩ PB)] = 1 – (0.6+0.8-0.5) = 0.1 -> 10 students
   - P(PA ∩ FB) = P(PA) – P(PA ∩ PB) = 0.6 – 0.5 = 0.1 -> 10 students
   - P(PB ∩ FA) = P(PB) – P(PB ∩ PA) = 0.8 – 0.5 = 0.3 -> 30 students

	PA	FA	B
PB	50	**30**	80
FB	**10**	**10**	**20**
A	60	**40**	100

(b) This is the conditional probability of PB given PA i.e. $P(PB|PA) = P(PA \cap PB)/P(PA) = 0.5/06 = 0.83$. Hence the possibility of passing B given the student already passed A is 83%.

(c) The two events:

(i) PA and PB are not mutually exclusive as $P(PA \cap PB) \neq 0$.

(ii) PA and PB are not independent either as $P(PB|PA) \neq P(PB)$. This is as $P(PB|PA) = P(PA \cap PB)/P(PA) = 0.5/0.6 = 0.83 \neq P(PB) = 0.8$.

5. Suppose you chose A out of the set of boxes A, B, and C to start with. Consider the two strategies, <u>stick</u> [S] or <u>change</u> [C].

   a. If the prize is in A, the host can open either box B or C and show it is empty. You win with S; sticking with box A and lose with C, changing to the box the host left unopened.
   b. If the prize is in B, the host has to open box C. You lose with S; win with C because you have to change to box B as box C is open.
   c. If the prize is in C, the host has to open box B. You lose with S; win with C because you change to box C, box B is open.

   Changing is the optimal strategy since you win 2 times out of three and the probability that the prize is in the other box is 2/3.

   This can also be shown by Bayes theorem. Let WA be the event that the prize is in box A (WB and WC for the remaining two options). Let HA be the event that the host opens box A (HB and HC for the remaining two options). Suppose you choose box A. The probability that you win the prize if you switch is the probability that the prize is in B and the host opened C plus the probability that the prize is in C and the host opened B:

$$P(WB \cap HC) + P(WC \cap HB) =$$
$$P(WB)P(HC|WB) + P(WC)P(HB|WC) =$$
$$1/3*1+1/3*1 = 2/3$$

   The second line follows from the definition of conditional probabilities: $P(A \cap B) = P(A|B)*P(B) = P(B|A)*P(A)$.